

# Heisenberg double versus deformed derivatives

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**Abstract.** A common replacement of the tangent space to a noncommutative space whose coordinate algebra is the enveloping algebra of a Lie algebra is generated by the deformed derivatives, usually defined by procedures involving orderings among noncommutative coordinates. We show that an approach to extending the noncommutative configuration space to a phase space, based on a variant of Heisenberg double, more familiar for some other algebras, e.g. quantum groups, is in the Lie algebra case equivalent to the approach via deformed derivatives. The dependence on the ordering is now in the form of the choice of a suitable linear isomorphism between the full algebraic dual of the enveloping algebra and a space of formal differential operators of infinite order.

**1.** Noncommutative algebras and noncommutative geometry may play various roles in models of mathematical physics; for example describing quantum symmetry algebras. A special case of interest is when the noncommutative algebra is playing the role of the space-time of the theory, and is interpreted as a small deformation of the 1-particle configuration space. If one wants to proceed toward developing field theory on such a space, it is beneficial to introduce the extension of the deformation of configuration space to a deformation of full phase space (symplectic manifold) of the theory. Deformed momentum space for the noncommutative configuration space whose coordinate algebra is the enveloping algebra of a finite-dimensional Lie algebra (also called Lie algebra type noncommutative spaces) has been studied recently in mathematical physics literature ([1, 2, 4]), mainly in special cases, most notably variants of so-called  $\kappa$ -Minkowski space ([2, 9, 8, 12]). We have related several approaches to the phase space deformations in [11], for a general Lie algebra type noncommutative space. The differential forms and exterior derivative can also be extended to the same setup ([15]).

The algebras in the article are over a field  $\mathbf{k}$  of characteristic zero; both real and complex numbers appear in applications of the present formalism. We fix a Lie algebra  $\mathfrak{g}$  with basis  $\hat{x}_1, \dots, \hat{x}_n$ .

**2.** Lie algebra type noncommutative spaces are simply the deformation quantizations of the linear Poisson structure; given structure constants  $C_{jk}^i$  linear in a deformation parameter the enveloping algebras of the Lie algebra  $\mathfrak{g}$  given in a base by  $[\hat{x}_j, \hat{x}_k] = C_{jk}^i \hat{x}_i$  is viewed as a deformation of the polynomial (symmetric) algebra  $S(\mathfrak{g})$ ; by  $x_i$  without hat we denote the generators of

that commutative algebra. Given any linear isomorphism  $\xi : S(\mathfrak{g}) \xrightarrow{\cong} U(\mathfrak{g})$  we transfer the noncommutative product on  $U(\mathfrak{g})$  to a  $\star$ -product on  $S(\mathfrak{g})$ , given by  $f \star g = \xi^{-1}(\xi(f) \cdot \xi(g))$ . There is a number of isomorphisms which may play role of  $\xi$ , but in order to introduce either deformed derivatives like in [1, 4, 12, 11], or to make the correspondence with the Heisenberg double construction, we need to restrict to  $\xi$  which are *coalgebra isomorphisms*; for small deformations we also require that  $\xi$  is the identity on the constant and linear parts, i.e. on  $\mathbf{k} \oplus \mathfrak{g} \subset S(\mathfrak{g})$ . Our restriction to coalgebra isomorphisms, singles out a distinguished class of star products quantizing the linear Poisson structure. Kathotia [6] compares the Kontsevich star product ([7]) for linear Poisson structures to the PBW-product which corresponds to the case where  $\xi$  is the standard symmetrization (coexponential) map; Kontsevich star product is not in our class, although it is equivalent to the PBW product, which is in our class.

**3. (Hopf actions and smash products)** Recall that a left action  $\triangleright : H \otimes A \rightarrow A$  of a Hopf algebra  $H$  on an algebra  $A$  is a Hopf action if it is satisfying the condition  $h \triangleright (a \cdot b) = \sum (h_{(1)} \triangleright a) \cdot (h_{(2)} \triangleright b)$ , where we used the Sweedler notation  $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ ; we also say that  $A$  is a left  $H$ -module algebra. In that case, one defines the smash product algebra (or crossed product)  $AH$  as the tensor product  $A \otimes H$  with the associative multiplication given by

$$(a \otimes h)(b \otimes g) = \sum (ah_{(1)} \triangleright b) \otimes (h_{(2)}g).$$

**4.** The input for the **Heisenberg double** construction is a pair of Hopf algebras  $H, H'$  in a bilinear pairing  $\langle, \rangle : H \otimes H' \rightarrow \mathbf{k}$  which is Hopf, i.e. with the product on pairings on the tensor square, the coproduct and the product are dual in the sense  $\langle \Delta_H(a), b \otimes c \rangle = \langle a, b \cdot c \rangle$ ,  $\langle a \otimes a', \Delta_{H'}b \rangle = \langle a \cdot a', b \rangle$  and similarly for the unit and counit. In our case  $H = U(\mathfrak{g})$  and the role of  $H'$  is played by the algebraic linear dual  $U^*(\mathfrak{g}) = \text{Hom}_{\mathbf{k}}(U(\mathfrak{g}), \mathbf{k})$  which is a *topological Hopf algebra*, i.e. the coproduct of the generators has infinitely many summands in the tensor square, amounting to the need for a (formal) completion of  $H \otimes H'$ . Similar to the Drinfel'd double, Heisenberg double is the algebra whose underlying space is (a completion of)  $H \otimes H'$ , but unlike Drinfel'd double it does not have a Hopf algebra structure itself. One defines an action of  $H'$  on  $H$  given by  $h' \triangleright h = \sum \langle h', h_{(2)} \rangle h_{(1)}$  where  $\Delta_H(h) = \sum h_{(1)} \otimes h_{(2)}$ ; as we required that the pairing is Hopf pairing, this action of  $H'$  on  $H$  is automatically a Hopf action (cf. 3), hence we can form

the corresponding smash product algebra  $H\sharp H'$ , the Heisenberg double of  $H$  (better, of the data  $(H, H', \langle, \rangle)$ ).

**5.** Coalgebra isomorphism  $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  induces a transpose map  $\xi^T : U^*(\mathfrak{g}) \rightarrow S^*(\mathfrak{g})$ , which is consequently an algebra isomorphism. There is an isomorphism between  $S^*(\mathfrak{g}) \cong \hat{S}(\mathfrak{g}^*)$  where  $\hat{S}(\mathfrak{g}^*)$  denotes a completed symmetric algebra on the dual; the isomorphism depends on a normalization of a pairing (cf. [3], **10.4**, **10.5**). Furthermore, functionals in  $S^*(\mathfrak{g}) \cong \hat{S}(\mathfrak{g}^*)$  can be identified with infinite order differential operators with constant coefficients: a differential operator applied to a polynomial in  $S(\mathfrak{g})$  and then evaluated at 0, gives rise to a differential operator. If the dual generators of  $\mathfrak{g}^* \subset \hat{S}(\mathfrak{g}^*)$  corresponding to the basis  $x_1, \dots, x_n$  are denoted as the partial derivatives  $\partial^i$ , this rule and identification explains the choice of normalization in [3]. The topological coproduct on  $U^*(\mathfrak{g})$  which is the algebraic transpose to the product on  $U(\mathfrak{g})$ , is (for  $\xi$  being the symmetrization map) written as a formal differential operators in  $\hat{S}(\mathfrak{g}^*)$  in [14], where the generalizations for Lie bialgebras are considered. In [3] we have shown that this coproduct is the same as a coproduct obtained by using Leibniz rules defined in terms of the deformed commutation relations; and in the case of symmetric ordering we have exhibited ([3]) a Feynman-like diagram expansion summing to basically a Fourier-transformed form of the BCH series.

**6.** A coalgebra isomorphism  $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  tautological on  $\mathbf{k} \oplus \mathfrak{g}$  as above, is equivalent (partly known in folklore, and described in detail in [11]) to any of several other data, including (the parameters for) certain exponential formulas ([4, 10, 13]) and  $\phi$ -data described as follows. The star product  $x_i \star f$  is always of the form  $\sum_j x_j \phi_j^i(\partial^1, \dots, \partial^n)(f)$  where  $(\phi_j^i)_{i,j=1,\dots,n}$  is a matrix of elements in  $\hat{S}(\mathfrak{g}^*)$  (formal power series in dual variables  $\partial^1, \dots, \partial^n$ ) satisfying a formal set of differential equations ([3] ch. 4) equivalent to the statement that the formula  $\phi(-\hat{x}_i)(\partial^j) = \phi_i^j$  defines a Lie algebra morphism  $\phi : \mathfrak{g} \rightarrow \text{Der}(\hat{S}(\mathfrak{g}^*))$ .

The correspondence  $\hat{x}_i \mapsto \hat{x}_j^\phi = \sum_j x_j \phi_i^j$  extends to an injective morphism of associative algebras  $U(\mathfrak{g}) \rightarrow \hat{A}_{n,\mathbf{k}}$  where  $\hat{A}_{n,\mathbf{k}}$  is the Weyl algebra of differential operators with polynomial coefficients, completed by the degree of the differential operator (hence we allow formal power series in  $\partial^i$ -s but not in  $x_j$ -s). This (semi)completed Weyl algebra has the standard Fock representation on  $S(\mathfrak{g})$ . The Lie algebra homomorphism  $\phi$  extends multiplicatively to a unique homomorphism  $U(\mathfrak{g}) \rightarrow \text{End}(\hat{S}(\mathfrak{g}^*))$  (also denoted  $\phi$ ), which is a Hopf action (3). Thus we can form a smash product algebra  $\hat{A}_{\mathfrak{g},\phi} = U(\mathfrak{g})\sharp\hat{S}(\mathfrak{g}^*)$ ,

the deformed Weyl algebra ( $\hat{A}_{n,\mathbf{k}}$  is a special case of this construction for the abelian Lie algebra). The rule  $\hat{x}_i \mapsto \hat{x}_i^\phi$ ,  $\partial^j \rightarrow \partial^j$  extends to a unique homomorphism  $\hat{A}_{\mathfrak{g},\phi} \rightarrow \hat{A}_{n,\mathbf{k}}$ ; one easily shows that it is an isomorphism.

**7.** Not only  $U(\mathfrak{g})$  acts by Hopf action on  $\hat{S}(\mathfrak{g}^*)$  (what was used in the construction of  $A_{\mathfrak{g},\phi}$ ), but also  $\hat{S}(\mathfrak{g}^*)$  as a topological Hopf algebra acts on  $U(\mathfrak{g})$ . The *latter* action is in the Main Theorem below identified with the smash product action of the Heisenberg double. To defined the latter action  $U(\mathfrak{g})$  is embedded as a subalgebra  $U(\mathfrak{g}) \# \mathbf{k} \hookrightarrow A_{\mathfrak{g},\phi}$ ; and similarly for  $\hat{S}(\mathfrak{g}^*)$ . The action  $a \otimes u \mapsto a \blacktriangleright \hat{u}$ ,  $A_{\mathfrak{g},\phi} \otimes U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  is defined by multiplying within  $A_{\mathfrak{g},\phi}$  and then projecting by evaluating the second tensor factor in  $A_{\mathfrak{g},\phi} = U(\mathfrak{g}) \# \hat{S}(\mathfrak{g}^*)$  (as a differential operator) at 1. Thus  $U(\mathfrak{g})$  is a  $A_{\mathfrak{g},\phi}$ -module, the deformed Fock space with  $1_{U(\mathfrak{g})}$  the deformed vacuum. If we define, for  $P \in \hat{S}(\mathfrak{g}^*) \hookrightarrow A_{\mathfrak{g},\phi}$ , the linear operator  $\hat{P} : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  by  $\hat{P}(\hat{u}) = \hat{P} \blacktriangleright \hat{u}$  or (equivalently by [11]) by  $\hat{P}(\xi(f)) = \xi(P(f))$ , then the Leibniz rule holds:  $\sum \hat{P}_{(1)}(\hat{u}) \cdot_{U(\mathfrak{g})} \hat{P}_{(2)}(\hat{v}) = P(\hat{u} \cdot_{U(\mathfrak{g})} \hat{v})$  for some deformed coproduct  $P \mapsto \sum P_{(1)} \otimes P_{(2)}$  on  $\hat{S}(\mathfrak{g}^*)$  (with the tensor product allowing infinitely many terms), cf. **13**. We mention ([11]) also that  $\xi : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  can be computed by composing  $S(\mathfrak{g}) \hookrightarrow A_{n,\mathbf{k}} \cong A_{\mathfrak{g},\phi} \xrightarrow{\blacktriangleright 1_{U(\mathfrak{g})}} U(\mathfrak{g})$ .

**8. Lemma.** (nonsymmetric formula for  $\Delta(\partial^\mu)$ , [11])

$$\Delta(\partial^\mu) = 1 \otimes \partial^\mu + \partial^\alpha \otimes [\partial^\mu, \hat{x}_\alpha] + \frac{1}{2!} \partial^{\alpha_1} \partial^{\alpha_2} \otimes [[\partial^\mu, \hat{x}_{\alpha_1}], \hat{x}_{\alpha_2}] + \dots \quad (1)$$

The sum has only finitely many terms when applied to an element in  $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ . Proof is by induction, see [11].

**9. Lemma.** If  $\hat{a} = a^\alpha \hat{x}_\alpha$  and  $\hat{f} \in U(\mathfrak{g})$  then

$$\hat{\partial}^\mu(\hat{a}^p \hat{f}) = \sum_{k=0}^{p-1} \binom{n}{k} a^{\alpha_1} a^{\alpha_2} \dots a^{\alpha_k} \hat{a}^{p-k} [[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots, \hat{x}_{\alpha_k}]](\hat{f}) \quad (2)$$

*Proof.* This is a tautology for  $p = 0$ . Suppose it holds for all  $p$  up to some  $p_0$ , and for all  $\hat{f}$ . Then set  $\hat{g} = \hat{a} \hat{f} = a^\alpha \hat{x}_\alpha$ . Then  $\hat{\partial}^\mu(\hat{a}^{p_0+1} \hat{f}) = \hat{\partial}^\mu(\hat{a}^{p_0} \hat{g})$  and we can apply (2) to  $\hat{\partial}^\mu(\hat{a}^{p_0} \hat{g})$ . Now

$$\begin{aligned} [[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots, \hat{x}_{\alpha_k}]](\hat{g}) &= a^{\alpha_k} [[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots, \hat{x}_{\alpha_k}]](\hat{x}_{\alpha_{k+1}} \hat{g}) \\ &= \hat{a} [[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots, \hat{x}_{\alpha_k}]](\hat{f}) + \\ &\quad + a^{\alpha_{k+1}} [[[\hat{\partial}^\mu, \hat{x}_{\alpha_1}], \dots, \hat{x}_{\alpha_k}], \hat{x}_{\alpha_{k+1}}]](\hat{f}). \end{aligned}$$

Collecting the terms and the Pascal triangle identity complete the induction step. It is interesting that this lemma was proved for calculational purposes in [11] and below in this article it will be seen as a step and a special case of a formula showing the Heisenberg double formula for an action coming out of deformed derivative approach.

**10. Theorem.** *Let  $\hat{u} \rightarrow \hat{u}^{\text{op}}$  be the algebra antiautomorphism of  $U(\mathfrak{g})$  extending the identity on  $\mathfrak{g}$ . Given a left Hopf action  $\phi : U(\mathfrak{g}) \rightarrow \text{End}(\hat{S}(\mathfrak{g}^*))$ , with  $\phi(-\hat{x}_i)(\partial^j) = \delta_j^i + O(\partial)$ , there is a Hopf pairing  $\langle, \rangle_\phi : U(\mathfrak{g}) \otimes \hat{S}(\mathfrak{g}^*) \rightarrow \mathbf{k}$  given by*

$$\langle \hat{u}, P \rangle_\phi = \phi(S_{U(\mathfrak{g})}\hat{u})(P)|0\rangle \equiv \phi(S_{U(\mathfrak{g})}\hat{u})(P)(1_{S(\mathfrak{g})}) \quad (3)$$

where  $\hat{u} \in U(\mathfrak{g})$ ,  $P \in \hat{S}(\mathfrak{g}^*)$ ,  $S_{U(\mathfrak{g})}$  is the antipode antiautomorphism of  $U(\mathfrak{g})$ , and where  $\hat{S}(\mathfrak{g}^*)$  is considered a topological Hopf algebra with respect to the  $\phi$ -deformed coproduct.

*Proof.* Clearly the pairing is well defined: the antipode comes because we use left Hopf actions. The product of differential operators *with constant coefficients* evaluated at 1 equals the product of evaluations at 1. Therefore the fact that  $\phi$  is Hopf implies  $\langle \hat{u}, PQ \rangle_\phi = \langle \Delta \hat{u}, P \otimes Q \rangle_\phi$ . It is only nonobvious to verify the other duality: of  $\phi$ -deformed coproduct and the multiplication on  $U(\mathfrak{g})$ . It is sufficient to show that one has

$$\langle \hat{x}_\alpha \hat{u}, \partial^\mu \rangle_\phi = \langle \hat{x}_\alpha \otimes \hat{u}, \Delta \partial^\mu \rangle_\phi. \quad (4)$$

for all  $\alpha$  and all  $\hat{u}$  in  $U(\mathfrak{g})$ . Indeed, extending to  $\prod_{i=1}^k x_{\alpha_i} \hat{u}$  for all  $(\alpha_1, \dots, \alpha_k)$  can be done by induction on  $k$ , using the coassociativity of the coproduct and associativity of the product. Once it is true for any products  $\hat{v}\hat{u}$  in the left argument, it is an easy general nonsense, using the already known duality for  $\Delta_{U(\mathfrak{g})}$ , to extend the property to products of  $\partial$ -s by induction using the following calculation for the induction step

$$\begin{aligned} \langle \hat{v}\hat{u}, P_1 P_2 \rangle_\phi &= \langle \sum \hat{v}_{(1)} \hat{u}_{(1)} \otimes \hat{v}_{(2)} \hat{u}_{(2)}, P_1 \otimes P_2 \rangle_\phi \\ &= \sum \langle \hat{v}_{(1)} \otimes \hat{u}_{(1)}, \Delta(P_1) \rangle_\phi \langle \hat{v}_{(2)} \otimes \hat{u}_{(2)}, \Delta(P_2) \rangle_\phi \\ &= \sum \langle \hat{v}_{(1)} \otimes \hat{u}_{(1)} \otimes \hat{v}_{(2)} \otimes \hat{u}_{(2)}, \Delta(P_1) \otimes \Delta(P_2) \rangle_\phi \\ &= \sum \langle \hat{v} \otimes \hat{u}, \Delta(P_1 P_2) \rangle_\phi \end{aligned}$$

Let us now calculate (4) using the nonsymmetric formula (1) for the  $\phi$ -coproduct. All terms readily give zero in first factor unless the first factor is degree 1 in  $\partial$ -s. Thus we effectively need to show

$$\sum_{\beta} \langle x_\alpha, \partial^\beta \rangle_\phi \otimes \langle \hat{u}, [\partial^\mu, \hat{x}_\beta] \rangle_\phi = \langle x_\alpha \hat{u}, \partial^\mu \rangle_\phi.$$

The left-hand side is  $\sum_{\beta} \phi(-\hat{x}_{\alpha})(\partial^{\beta})\phi(S_{U(\mathfrak{g})}\hat{u}^{\text{op}})(\phi(-\hat{x}_{\beta})(\partial^{\mu}))|0\rangle = \sum_{\beta} \phi(-\hat{x}_{\alpha})(\partial^{\beta})|0\rangle\phi(S_{U(\mathfrak{g})}(\hat{u}^{\text{op}}\hat{x}_{\beta})(\partial^{\mu}))|0\rangle$  and  $\phi(-\hat{x}_{\alpha})(\partial^{\beta})|0\rangle = \delta_{\alpha}^{\beta}$  by the assumption on  $\phi$ . The contraction with the Kronecker gives  $\phi(S_{U(\mathfrak{g})}(\hat{x}_{\alpha}\hat{u})^{\text{op}})(\partial^{\mu})|0\rangle$ .

**11. Proposition.** *If  $\xi = \xi_{\phi} : S(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  is the coalgebra isomorphism corresponding to  $\phi$  and  $\xi^T : U(\mathfrak{g})^* \rightarrow S(\mathfrak{g})^* \cong \hat{S}(\mathfrak{g}^*)$  its transpose, then the pairing may be described alternatively by*

$$\langle \hat{u}, P \rangle_{\phi} = (\xi^T)^{-1}(P)(\hat{u}) = P(\xi_{\phi}^{-1}(\hat{u})) = \epsilon_{S(\mathfrak{g})}(\hat{u}^{\phi}|0\rangle) = \epsilon_{S(\mathfrak{g})}((\hat{P}(\hat{u}))^{\phi}|0\rangle), \quad (5)$$

where the evaluation  $P(\xi_{\phi}^{-1}(\hat{u}))$  is by the standard pairing of  $\hat{S}(\mathfrak{g}^*)$  with  $S(\mathfrak{g})$ .

*Proof.* It is only left to show that the alternative formula (5) gives the same (and, in particular, Hopf) pairing. By the previous arguments, it is sufficient to show this when  $P = \partial^{\mu}$ . This is evident when  $\hat{u} = \hat{x}_{\nu}$  for some  $\nu$ . Now suppose by induction that (5) holds for  $\hat{u}$ . Then

$$\begin{aligned} \epsilon \partial^{\mu}(\hat{x}_{\lambda}^{\phi}\hat{u}^{\phi}|0\rangle) &= \epsilon \partial^{\mu} x_{\alpha} \phi_{\lambda}^{\alpha} \hat{u}^{\phi}|0\rangle \\ &= \epsilon x_{\alpha} \partial^{\mu} \phi_{\lambda}^{\alpha} \hat{u}^{\phi}|0\rangle + \epsilon \phi_{\lambda}^{\mu} \hat{u}^{\phi}|0\rangle \\ &= 0 + \epsilon \phi(S_{U(\mathfrak{g})}\hat{u})([\partial^{\mu}, \hat{x}_{\lambda}]) \\ &= \epsilon \phi(S_{U(\mathfrak{g})}(\hat{x}_{\lambda}\hat{u}))(\partial^{\mu}), \end{aligned}$$

hence it holds for  $\hat{x}_{\lambda}\hat{u}$ .

**12. Main Theorem.** *The  $(\mathfrak{g}, \phi)$ -twisted Weyl algebra  $A_{\mathfrak{g}, \phi}$  is isomorphic to the Heisenberg double for the pair of the Hopf algebra  $U(\mathfrak{g})$  and the topological Hopf algebra  $\hat{S}(\mathfrak{g}^*)$  with respect to the  $\phi$ -deformed coproduct, and for the Hopf pairing which is given above. In other words, the left action  $\blacktriangleright$  used for the second smash product structure satisfies (and is determined by) the formula*

$$P \blacktriangleright \hat{u} = \sum \langle \hat{u}_{(2)}, P \rangle_{\phi} \hat{u}_{(1)}$$

for all  $\hat{u} \in U(\mathfrak{g})$  and  $P \in \hat{S}(\mathfrak{g}^*)$ .

*Proof.* If the identity holds for  $P = P_1$  and  $P = P_2$  then

$$\begin{aligned} P_1 P_2 \blacktriangleright \hat{u} &= P_1 \blacktriangleright \left( \sum \langle \hat{u}_{(2)}, P_2 \rangle_{\phi} \hat{u}_{(1)} \right) \\ &= \sum \langle \hat{u}_{(3)}, P_2 \rangle_{\phi} \langle \hat{u}_{(2)}, P_1 \rangle_{\phi} \hat{u}_{(1)} \\ &= \sum \langle \hat{u}_{(2)}, P_1 P_2 \rangle_{\phi} \hat{u}_{(1)} \end{aligned}$$

hence it holds for  $P = P_1 P_2$ . As  $P = 1$  is trivial, it is hence sufficient to check  $P = \partial^{\mu}$ . The identity is linear in  $\hat{u} \in U(\mathfrak{g})$ , so it is sufficient to prove

it for all  $\hat{u}$  of the form  $\hat{u} = \hat{a}^p = (\sum_{\alpha=1}^n a^\alpha \hat{x}_\alpha)^p$ ,  $p \geq 0$  where  $\hat{a} = \sum_\alpha a^\alpha \hat{x}_\alpha$  is arbitrary. In that case,  $\sum u_{(2)} \otimes u_{(1)} = \sum_{k=0}^p \binom{n}{k} \hat{a}^k \otimes \hat{a}^{p-k}$  and we need to show

$$\partial^\mu \blacktriangleright \hat{a}^p = \hat{\partial}^\mu(\hat{a}^p) = \sum_{k=0}^p \binom{n}{k} \langle \hat{a}^k, \partial^\mu \rangle_\phi \hat{a}^{n-k}$$

but  $\langle \hat{a}^k, \partial^\mu \rangle_\phi$  is by (5) equal to

$$\phi(S_{U(\mathfrak{g})}(\hat{a}^k))(\partial^\mu) = (-1)^k \phi(\hat{a}^k)(\partial^\mu) = (-1)^k [\dots [[\partial^\mu, \hat{a}], \hat{a}], \dots, \hat{a}],$$

what by linearity reduces to (2) for the case  $f = 1$ .

One can easily compute that if  $[\partial, \hat{x}] = Q \in \hat{S}(\mathfrak{g}^*)$ , then also  $[\hat{\partial}, \hat{x}] = \hat{Q}$  for  $\partial \in \mathfrak{g}^*$ . Therefore for the generators, the commutation relations in the two smash products agree, hence the isomorphism.

**13. Remark.** The fact that the deformed coproduct defined by the deformed Leibniz rule for the action of  $\hat{S}(\mathfrak{g}^*)$  on  $U(\mathfrak{g})$  (for any  $\mathfrak{g}$  and  $\phi$ ) is well-defined coassociative map in the deformed derivative picture ([1, 2, 12, 11]) is obvious up to an ambiguity by an operator in the tensor square which is zero except on the kernel of the multiplication map in  $U(\mathfrak{g})$ . But now the Hopf action is well-defined within the Heisenberg double construction and the Heisenberg double as an algebra is identified with  $A_{\mathfrak{g}, \phi}$  where the deformed Leibniz rule is originally defined. Heisenberg double is an invariant picture, giving simple "dual" interpretation to the deformed coproduct, while the approach via deformed derivatives and commutators is useful for calculation, as most of the physics literature on the subject exhibited before.

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## References

- [1] G. AMELINO-CAMELIA, M. ARZANO, *Coproduct and star product in field theories on Lie-algebra non-commutative space-times*, Phys. Rev. D65:084044 (2002) [hep-th/0105120](#).
- [2] M. DIMITRIJEVIC, F. MEYER, L. MÖLLER, J. WESS, *Gauge theories on the kappa-Minkowski spacetime*, Eur.Phys.J. C36 (2004) 117–126; [hep-th/0310116](#).

- [3] N. DUROV, S. MELJANAC, A. SAMSAROV, Z. ŠKODA, *A universal formula for representing Lie algebra generators as formal power series with coefficients in the Weyl algebra*, Journal of Algebra 309, Issue 1, pp.318–359 (2007) (math.RT/0604096)
- [4] S. HALLIDAY, R. J. SZABO, *Noncommutative field theory on homogeneous gravitational waves*, J. Phys. A39 (2006) 5189–5226, arXiv:hep-th/0602036.
- [5] M. KAPRANOV, *Heisenberg doubles and derived categories*, J. Alg. 202, 712–744 (1998), arXiv:q-alg/9701009.
- [6] V. KATHOTIA, *Kontsevich’s universal formula for deformation quantization and the Campbell-Baker-Hausdorff formula*, Internat. J. Math. 11 (2000), no. 4, 523–551; math.QA/9811174.
- [7] M. KONTSEVICH, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. 66 (2003), no. 3, 157–216.
- [8] S. KREŠIĆ-JURIĆ, S. MELJANAC, M. STOJIC, *Covariant realizations of kappa-deformed space*, hep-th/0702215.
- [9] J. LUKIERSKI, A. NOWICKI, *Heisenberg double description of  $\kappa$ -Poincaré algebra and  $\kappa$ -deformed phase space*, arXiv:q-alg/9792003.
- [10] S. MELJANAC, A. SAMSAROV, M. STOJIC, K. S. GUPTA, *Kappa-Minkowski space-time and the star product realizations*, Eur.Phys.J. C51 (2007) 229–240 arXiv:0705.2471.
- [11] S. MELJANAC, Z. ŠKODA, *Leibniz rules for enveloping algebras*, arXiv:0711.0149; with newer version at <http://www.irb.hr/korisnici/zskoda/scopr5.pdf>.
- [12] S. MELJANAC, M. STOJIC, *New realizations of Lie algebra kappa-deformed Euclidean space*, Eur.Phys.J. C47 (2006) 531–539; hep-th/0605133.
- [13] S. MELJANAC, D. SVRTAN, Z. ŠKODA, *Exponential formulas and Lie algebra type star products*, preliminary version.
- [14] N. RESHETIKHIN, *Quantization of Lie bialgebras*, Int. Math. Res. Notices, 1992, no.7, 143–151.
- [15] Z. ŠKODA, *Twisted exterior derivative for enveloping algebras*, arXiv:0806.0978.